Chapter 2: Discrete Random Variables

Content:

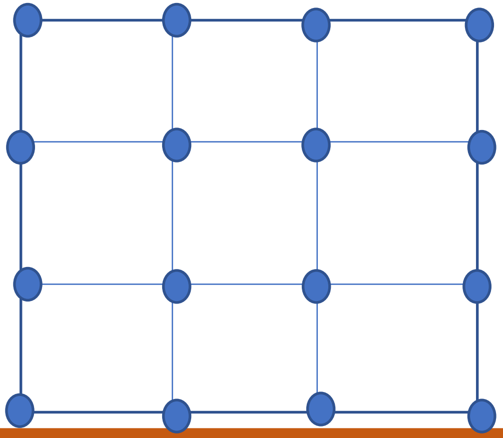
1. Random Variable
2. Probability Mass Function (PMF)
3. Known probability Mass Functions
4. Functions of Random Variables
5. Expectation/Mean, Variance, Standard Deviation, Moment
6. Joint PMFs of Multiple Random Variables
7. Conditioning
8 . Independence

Random Variables

- **Random Variable**

A **random variable** a is a measurable **function** $X: \Omega \rightarrow E$, Ω as the set of possible outcomes and E a measurable space.

- **Remark:**
- In this course, we are interested in real-valued functions, as a consequence, the set E is $\mathbb{R}$ or a subset of $\mathbb{R}$.
- **Example:**
- The experiment of rolling a die twice. We can define the function « maximum of the two rolls » as a random variable.
- In the exepriment of tossing a coin. The function that assigns the value « 0 » to the result of getting a Head, and « 1 » to the result of getting a tail is a random variable.



Random Variable

- A random variable is **a real-valued function** of the outcome of the experiment.
- A random variable can be **conditioned** on an **event** or on another **random variables**.
- There is a notion of **independence** of a random variable from an event or from another random variable

Discrete Random Variable

- A random variable is **discrete** if its range is either **finite** or **countably infinite**.

Notation

- In general, we will note a random variable using a **capital letter** such as **X**, and any **possible value** of X will be denoted **lower-case x**.

Probability Mass Function

- **Probability Mass Function**

- A discrete random variable is characterized by a probability mass function, also called shortly PMF, and denoted p_X.

$$p_X(x) = P(\{X = x\})$$

- PMF is the probability of the event $\{X = x\}$ consisting of all outcomes that give rise to a value of X equal to x.
- Note that **(Very Important)**

$$\sum_{x \in E} p_X(x) = 1$$

- x ranges over all possible values of X, and **the events $\{X = x\}$ are distjoints** and **form a partition of the sample space.**

- **Notations**

- We will usually write $P(X = x)$ instead of $P(\{X = x\})$.
- To denote the probability that a random variable X takes a value in a set S, we will use $P(X \in S)$.

- **Example:**

- Consider the experiment of tossing a fair coin twice. Let X be the number of heads obtained. Then the

PMF of X is $p_X(x)\begin{cases} \frac{1}{4} if\ x = 0\ or\ x = 2 \\ \frac{1}{2} if\ x = 1 \\ 0\ otherwise \end{cases}$

- Express $P(X \in S)$ using $p_X(x)$ where $S \subseteq \Omega$

Examples of known probability mass functions

- Examples of known probability mass functions

- **Bernoulli random variable**
- The bernoulli random variable models the success/fail of an experiment. It takes two possible outcomes: 1 if the experiment is a success or 0 if it is a fail.

$$X = \begin{cases} 1 \text{ } if \text{ } success \\ 0 \text{ } if \text{ } fail \end{cases}$$

- The PMF of a Bernoulli random variable is

$$p_X(x) = \begin{cases} p & if \ x = 1 \\ 1 - p & if \ x = 0 \end{cases}$$

- We will say that X is a **Bernoulli variable with parameter p**.
- In general, Bernoulli random variable is used to model experiments with only two possible outcomes.

- Example of experiments that can be modeled using Bernoulli random variable

- The state of a telephone at a given time that can be either free or busy.
- A person who can be either healthy or sick.
- The preference of a person who can be either for or against a certain political candidate.
- A toss of a coin ☺.

- **Examples of known probability mass functions**

- **Binomial Random Variable**
- X is a Binomial variable of parameters **n** and **p** if the PMF of X consists of the binomial probabilities such that

$$p_X(k) = P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \text{ , for } k = 0, 1, \ldots, n$$

- Here, we use k instead of x for convenience, and to emphasize that the outcome is an integer number.

- **Examples of known probability mass functions**

- **Geometric Random Variable**
- Suppose we repeatedly and independently toss a coin with probability of head equal to p, where $0 < p < 1$. The geometric random variable is **the number X of tosses needed for a head to come up for the first time**. Its PMF is given by

$$\boldsymbol{p_X(k) = (1-p)^{k-1}p,\ \ k = 1, 2, ..}$$

- **Exercise:**

Why this is a legitimate PMF?

- **Examples of known probability mass functions**

- **Uniform Random Variable**
- A uniform random variable has a PMF given by

$$\boldsymbol{p_X(k) = \frac{1}{N}}, \ k = 0,1,2,..,N-1$$

where N is a strictly positive integer parameter characterizing the PMF.

- **Example:**

- Rolling a fair die once follows a uniform distribution

- **Examples of known probability mass functions**

- **Poisson Random Variable**
- A poisson random variable has a PMF given by

$$\boldsymbol{p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}},\ k = 0,1,2,..$$

where $\boldsymbol{\lambda}$ is a positive parameter characterizing the PMF.

- **Example:**

In general, Poisson distribution is used to predict or explain the number of events occurring within a given interval of time or space

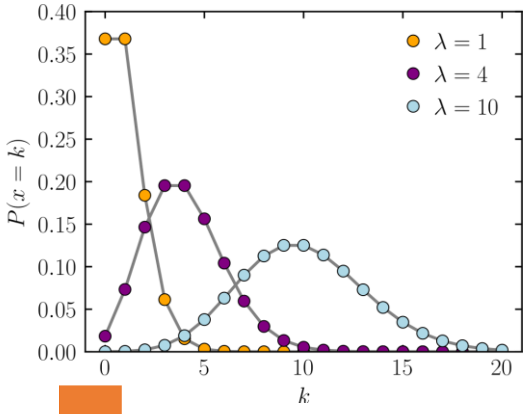
- **Exercise:**
- Why this is a legitimate PMF?

The histogram of a random variable:

- The histogram of a random variable measures the number of occurrence of an event. It gives a general idea on the probability mass function.

Exercise:

Match the histograms to the PMF



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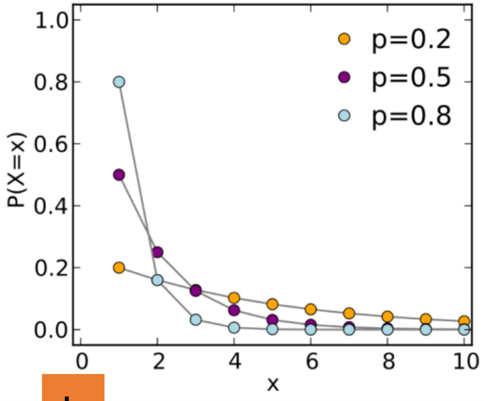

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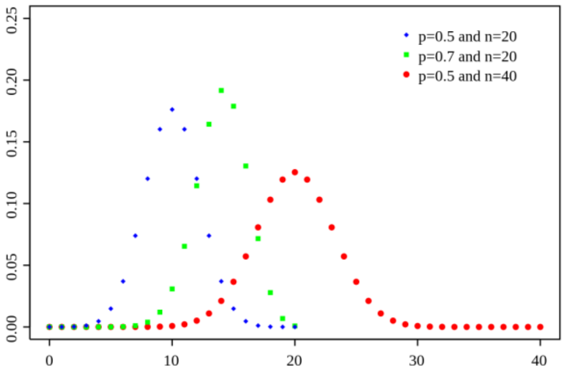

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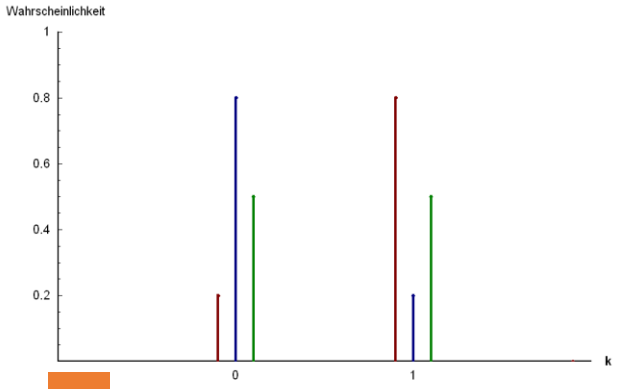


d

1

$$p_X(x) = \begin{cases} p & if\ x = 1 \\ 1-p & if\ x = 0 \end{cases}$$

2

$$p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!},$$

$$k = 0,1,..$$

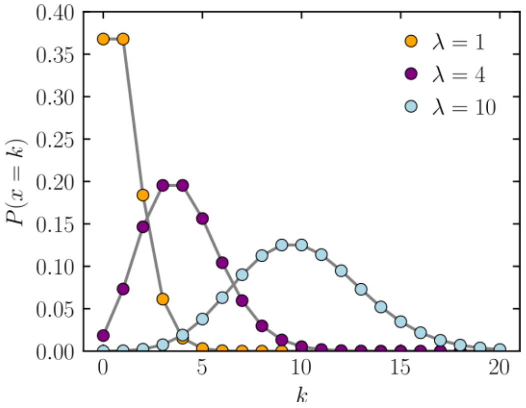
3

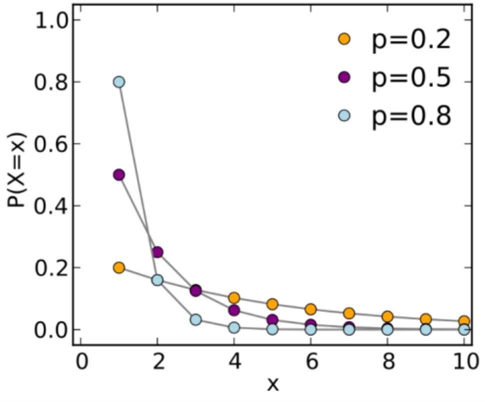
$$p_X(k) = (1-p)^{k-1}p,$$
$$k = 1,2,..$$

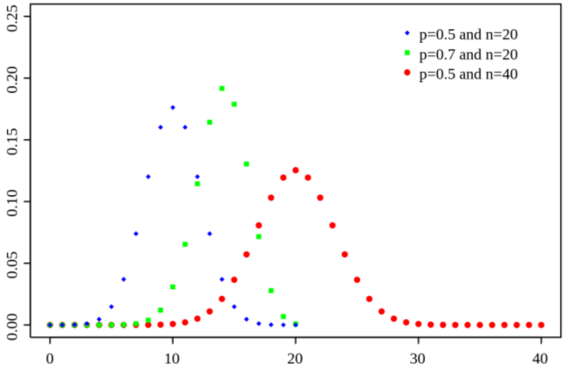
4

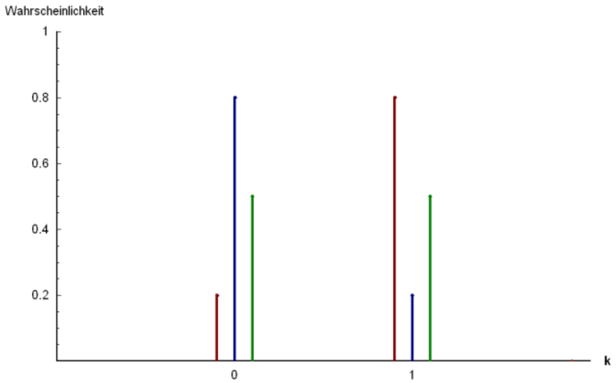
$$p_X(k) = P(X=k) = \binom{n}{k}p^k(1-p)^{n-k}$$
$$k = 0,1,..,n$$

The histogram of a random variable:









Assignment (to be uploaded on Canevas on Monday March 6th):

- Propose a code to generate the histograms of Bernoulli, Binomial, Geometric, and Poisson distribution based only on a uniform random variable.
- Draw the histogram of the 4 distributions , using your code.
- Find a ready to use commands to plot the histogram of the 4 random variables. And draw the histograms using these commands.
- Show that the poisson PMF with parameter $\lambda = np$, with n very large and p very small can be approximated with a Binomial PMF.
- Compute the Poisson PMF for k=5, and compare when using a Binomial with n=100 and p=0.01.

Functions of Random Variables

Functions of Random Variables

- We consider X a random variable, let $g: \Omega \to \mathbb{R}$ be a function , we use the representation $\mathrm{Y} = g(X)$ to define a new random variable which outcomes are given by $g(x)$, in particular

$$p_Y(y) = \sum_{\{x|g(x)=y\}} p_X(x) .$$

- **Example:**

- Let $Y = |X|$ and let us apply the preceeding formula for the PMF $p_Y(y)$ to the case where

$$p_X(x) = \begin{cases} \frac{1}{9} & if\ x = -4,-3,-2,-1,0,1,2,3,4 \\ 0 & otherwise. \end{cases}$$

What are the possible values of Y? and what is $p_Y(y)$?